Fast Solver for Sum of Two Kronecker Products

- Find U_A, V_A such that $U_A A_1 V_A = D_1$, $U_A A_2 V_A = D_2$ are diagonal. Similarly $U_B B_1 V_B = D_3$, $U_B B_2 V_B = D_4$ are diagonal.

$$
\begin{aligned}
b &= (A_1 \otimes B_1 + A_2 \otimes B_2)\, x && (1)\\
(U_A \otimes U_B)\, b &= (U_A \otimes U_B)\,(A_1 \otimes B_1 + A_2 \otimes B_2)\, x && (2)\\
\tilde{b} &= (D_1 \otimes D_3 + D_2 \otimes D_4)\, \tilde{x} && (3)\\
\tilde{b} &= (U_A \otimes U_B)\, b, \quad x = (V_A \otimes V_B)\, \tilde{x} && (4)
\end{aligned}
$$

QZ Decomposition

- $QZ(A, B)$ generalized eigen decomposition to produce eigenvectors V, triangular $\mathcal{A}$, $\mathcal{B}$, and orthogonal Q, Z, such that $\alpha = \text{diag}(\mathcal{A})$, $\beta = \text{diag}(\mathcal{B})$,

$$\mathcal{A} = QAZ, \quad \mathcal{B} = QBZ, \quad AV\,\text{diag}(\beta) = BV\,\text{diag}(\alpha)$$

- Let $U = \tilde{U}^{-1}$ where

$$\tilde{U}(:, i) = \begin{cases} AV(:, i) & \text{if } |\alpha_i| \geq |\beta_i| \\ BV(:, i) & \text{otherwise} \end{cases}$$

  then UAV is diagonal, $U(BV\,\text{diag}(\alpha)) = U(AV\,\text{diag}(\beta))$ is also diagonal.

- Problem if V (or $\tilde{U}$) is not full rank.

Nearest Kronecker Product

- Given $mn \times mn$ matrix C, find matrices $A(m \times m)$, $B(n \times n)$ to minimize $\|C - A \otimes B\|_F$.
- Want $C([ib, \mathbf{ia}], [jb, \mathbf{ja}]) \approx A(\mathbf{ia}, \mathbf{ja})B(ib, jb)$ (`reshape(C,[n,m,n,m])`), let $mm \times nn$ matrix $\tilde{C}$ be (`permute(C,[2,4,1,3])`),

$$\tilde{C}([\mathbf{ia}, \mathbf{ja}], [ib, jb]) = C([ib, \mathbf{ia}], [jb, \mathbf{ja}]) = A(\mathbf{ia}, \mathbf{ja})B(ib, jb)$$

- SVD of $\tilde{C}$, left and right singular vectors reshaped to $A(ia, ja)$ and $B(ib, jb)$,

$$\tilde{C} \approx \sigma_1 \text{vec}(U_1)\text{vec}(V_1)' + \ldots + \sigma_k \text{vec}(U_k)\text{vec}(V_k)'$$

$$A(ia, ja) = \sigma_1 U_1([ia, ja]), \quad B(ib, jb) = V_1([ib, jb]) \ .$$

Complex Product

- Consider complex product,

$$(\mu_1 A_1 + i\mu_2 A_2) \otimes ((1/\mu_1)B_1 - i(1/\mu_2)B_2) = A_r + iA_c$$

  where real part $A_r = A_1 \otimes B_1 + A_2 \otimes B_2$, complex part $A_c = (\mu_2/\mu_1)A_2 \otimes B_1 - (\mu_1/\mu_2)A_1 \otimes B_2$.

- Choice of $\mu_1^2 = \|A_2 \otimes B_1\|_F$, $\mu_2^2 = \|A_1 \otimes B_2\|_F$ to approximately reduce $\|A_c\|_F$

- Sum of 4 real Kronecker Products can be computed as (real part of) sum of 2 complex Kronecker Products.

- Fast solver via QZ for complex matrix when interested only in the real part.

Questions (I need help)

- In NKP ($\|C - \sum_j A_j \otimes B_j\|$), can we impose special properties or sparsity patterns on A_j or B_j, e.g. require A_j be tridiagonal or B_j be diagonal?
- If $\sum_j A_j \otimes B_j$ (from NKP) is used as a preconditioner for solving $Cx = b$, would $\sum_j A_j \otimes B_j$ be ill-conditioned?
- What can be done if QZ returns rank deficient "V" (matrix of generalized eigenvectors).
- Can 3 (or more) matrices be simultaneously reduced to triangular form or some "convenient" form? This would be helpful in a solver for sum of 3 or more kronecker products.
- If we can easily solve a complex matrix $(A_r + iA_c)$, can we also easily solve with the real part A_r?

Kronecker Product

- Let matrices A be $mA \times nA$ and B be $mB \times nB$ be indexed as $A(ia, ja)$ and $B(ib, jb)$. Let $C = A \otimes B$ (or kron(A, B) in MATLAB notation), then matrix C is size $(mA * mB) \times (nA * nB)$.
- If matrix A is 3×3, then

$$C = \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \\ a_{31}B & a_{32}B & a_{33}B \end{bmatrix}.$$

- Matrix C can be viewed as a 4-index array $C([\mathbf{ib}, ia], [\mathbf{jb}, ja]) = A(ia, ja) * B(\mathbf{ib}, \mathbf{jb})$, where the composite index $[ib, ia] = ib + (ia - 1) * mB$ is the index in Fortran column-wise order.