Pierre Comon

Tensor Canonical Decomposition (CAND)

$$\boldsymbol{T} = \sum_{i=1}^{r} \boldsymbol{u}(i) \circ \boldsymbol{v}(i) \circ \ldots \boldsymbol{w}(i) \tag{1}$$

- r = "tensor rank"
- Related to PARAFAC
- r is a function of (order,sizes), and may exceed all sizes
- CAND holds on a ring, eg. integers (no need of a field)
- If $\mathcal{L}$ denotes an invertible change of coordinate:

  $$\mathcal{L}(CAND(\boldsymbol{T})) = CAND(\mathcal{L}(\boldsymbol{T}))$$

  which justifies the word "tensor". In other words: CAND (and a fortiori the tensor rank) is invariant under $\mathcal{L}$

Orthogonal Array decomposition (HOSVD)

$$\boldsymbol{T} = \boldsymbol{C} \underset{1}{\bullet} \boldsymbol{U} \underset{2}{\bullet} \boldsymbol{V} \underset{3}{\bullet} \ldots \underset{d}{\bullet} \boldsymbol{W} \tag{2}$$

- Related to Tuker3
- NOT RELATED to “tensor rank” (defined by CAND), but to $n-$mode ranks.

  These are linked only through inequalities.
- HOSVD not invariant by linear invertible change of coordinates:

  $$\mathcal{L}(CAND(\boldsymbol{T})) = CAND(\mathcal{L}(\boldsymbol{T}))$$

  The change needs to be orthogonal.

Approximations

- Independent Component Analysis (ICA): this is a SYMMETRIC CAND

  Also often (but not always) understood under the restriction:

  number of rank-one terms $\leq$ size

- PARAFAC: uniqueness proved under a constraint:

number of rank-one terms $\leq$ given function(size,order)

- Low $m-$mode rank: this is ANOTHER approximation.

  The (nondiagonal) core tensor is approximated by another of smaller size.

Generic rank

$$\mathcal{Z}_r = \{\boldsymbol{T} : \; rank(\boldsymbol{T}) = r\}$$

$$\mathcal{Y}_r = \{\boldsymbol{T} : \; rank(\boldsymbol{T}) \leq r\}$$

- $\mathcal{Y}_{r-1} \subset \mathcal{Y}_r$
- $\mathcal{Z}$ and $\mathcal{Y}$ not always closed
- $\mathcal{Y}_1 \subset \mathcal{Y}_2 \subset \ldots \subset \mathcal{Y}_{\bar{R}} = \mathcal{Y}_{\bar{R}+1} = \ldots$
  defines $\bar{R}$ = GENERIC RANK in space of tensors of given sizes and order
- In general: min(size) < generic rank < maximal rank
- Closure $\bar{\mathcal{Z}}_{\bar{R}} = \bar{\mathcal{Y}}_{\bar{R}}$

Dependence on underlying ring

- For any real tensor $\boldsymbol{T} \in \mathcal{T}[\mathbb{R}]$
  Denote
  CAND in $\mathbb{R}$: $rank_{\mathbb{R}}(\boldsymbol{T})$
  CAND in $\mathbb{C}$: $rank_{\mathbb{C}}(\boldsymbol{T})$
  Then
  $$rank_{\mathbb{C}}(\boldsymbol{T}) \leq rank_{\mathbb{R}}(\boldsymbol{T})$$
- For any completely symmetric tensor $\boldsymbol{S} \in \mathcal{S}$, we have similarly
  $$rank_{\mathcal{T}}(\boldsymbol{S}) \leq rank_{\mathcal{S}}(\boldsymbol{S})$$

Uniqueness

- Uniqueness not guaranteed in general for generic ranks or higher.
- For strictly smaller ranks, uniqueness ensured in general (except for a small subset of tensors).
- PARAFAC algorithms fall in the latter case